AN INTRODUCTION TO THE APPLICATION OF

LINEAR PROGRAMMING TO INDUSTRIAL ENGINEERING

A Thesis

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I wish to express my gratitude to Professor James H. Greene whose interest in this field prompted me to undertake this study, and whose encouragement and assistance benefited me greatly.

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ABSTRACT

Doty, William, K., An Introduction to the Application of Linear Programming to Industrial Engineering, May, 1954. There are 50 pages, 5 figures, and 10 references in the bibliography.

The purpose of this paper is to explain what linear programming is, how the linear programming problem is interpreted mathematically, how to solve a simple problem, what the assumptions, limitations, and possibilities of this technique are, and how to set up for solution a few examples of feasible problems. The emphasis is on Industrial Engineering applications and requires no knowledge of mathematics beyond college algebra.

The mathematical discussion illustrates the necessary elementary vector algebra including addition of vectors, unit vectors, and linearly independent sets. Linear transformations are illustrated by transformations from a 3-dimensional to a 2-dimensional space. Convex sets and convex polyhedral cones are illustrated. A simple numerical example with three unknowns in two equations,with a linear functional to be maximized, is solved step by step and the geometrical significance of each step is explained.

The four assumptions necessary for the application of linear programming are discussed and certain feasible problem areas in Industrial Engineering are listed. It is shown why the technique cannot be used for scheduling, sequence of production selection, elimination of expediters, etc.

Three mathematical models are presented: (1) machine assignment to maximize total productivity, (2) production line loading for a single

product to meet sales requirements over successive time periods and to minimize the combined costs of regular and overtime production and storage, and (3) a numerical example to show the complexity to which even problems of small size are subject. This model deals with three products, two production operations, three time periods, and the maximization of profit.

The conclusion points out that the application of linear programming to Industrial Engineering is, at present, limited, but predicts that, as study in both theory and practical aspects continues, more and more Industrial Engineering application will be made in the future.

AN INTRODUCTION TO THE APPLICATION OF LINEAR PROGRAMMING TO INDUSTRIAL ENGINEERING

INTRODUCTION

Considerable interest has developed in the past few years in the mathematical technique known as linear programming. This interest, at first, centered itself among mathematicians and economists because the concepts were similar to or were advancements in technique under study in those two sciences. Interest in other fields was limited until Dr. George B. Dantzig[1] evolved the "simplex method" of computation. This was a turning point. The formerly theoretical and limited aspects of linear programming were now found to have direct application to practical problems facing the armed forces[2], industry[3], agriculture[4] and transportation[5]. Evidence of widening interest is illustrated by the appearance of articles on linear programming in such practical minded publications as Factory Management & Maintenance[6] and Business

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1. Dantzig, G.B., "Maximization of a Linear Function of Variables Subject to Linear Inequalities," Activity Analysis of Production and Allocation, Cowles Commission Monograph 13; New York, John Wiley & Sons, 1951. pp 339-347.
2. Wood, M.K., and Geisler, M.A., "Development of Dynamic Models for Program Planning," (ibid.), pp. 189-215.
3. Charnes, A., Cooper, W.W., and Mellon, B., "Blending Aviation Gasolines," Econometrica, vol. 20, p. 2.
4. Hildreth, C., and Reiter, S., "On the Choice of a Crop Rotation Plan," Activity Analysis of Production and Allocation, Cowles Commission Monograph 13; New York, John Wiley & Sons, 1951, pp. 177-188.
5. Dantzig, G.B., "Application of the Simplex Method to a Transportation Problem," (ibid.), pp. 359-373.
6. Melden, M.E., "Operations Research," Factory Management & Maintenance, October 1953, vol. 111, No. 10, pp. 113-120.

Week[7]. In addition, the success of Operations Research and the attendent publicity has helped to focus attention to linear programming because this mathematical technique is a useful tool for Operations Research teams.

7. "Top Management by Mathematics," Business Week, May 30, 1953, No. 1239, pp. 86–92.

PURPOSE

The majority of the literature to date has been slanted to the field of economics and mathematics and hence requires a certain degree of mathematical maturity for its understanding. A few articles appearing in periodicals have intimated the possibilities of application of linear programming to Industrial Engineering problems. Between the two approaches lies a wide gap.

The purpose of this thesis then is to attempt to bridge this gap by explaining what linear programming is, how the linear programming problem is interpreted mathematically, how to solve a simple problem, what the assumptions, limitations, and possibilities, of this technique are, and how to set up for solution a few examples of feasible problems. The present treatment will endeavor to cover the material without requiring any mathematics beyond college algebra.

PRELIMINARY DISCUSSION

Linear programming is a mathematical technique which may be utilized in the optimal planning of activities which are interdependent. Programming in the title indicates the area of interest, the preparation of plans, which, when executed, will yield the best results. Programming is apparent in such areas, for instance, as mixtures of chemicals, loading of machine tools, assignment of personnel, and shipping allocation.

By linear is meant the restriction that the variables or unknowns must occur to the first power. No squares, cubes, etc., are permissable, nor may one variable be multiplied by another. Best or optimal results may refer, for example, to maximizing profits, or production, or to minimizing cost of production or storage.

The linear programming technique is carried out as follows:

1. A mathematical model (or set of equations) is formulated from the word problem, this set of equations to follow a certain form.
2. The mathematical model is solved,usually by the simplex method.

In regard to the formulation of the mathematical model it need not apply that the equations in the model express rigorously every facet, every fine point which could conceivably affect the problem at hand. Assumptions and approximations may be necessary. "Standard time", a statistical approximation, for example, may be employed in linear programming. Simplification of the problem may be advantageous, if not necessary. A certain phase of a production activity, for example, may be ignored if its effect on the total is small. When formu-

lating the model, therefore, assumptions, approximations and simplifications may be made provided the model retains enough predictive value to be useful.

In the construction of the mathematical model there are two methods of approach. In the majority of the literature which, as has been stated, has evolved from an economics background where the interest lay in covering an entire economy, large formalized models of economic activities have evolved[8]. This activity analysis approach, as it is called, has been extended to the theory of the firm so that many concrete and every day problems of industry may be investigated[9].

The second method of approach to the construction of the model, which is the one advanced by Charnes, Cooper and Henderson[10], is "to approach each problem squarely, forming the model and calculational procedures naturally from the demands of the material." They feel that the activity analysis approach is awkward and artificial and "carries with it the danger of forcing the problem or overlooking important aspects which do not fit easily into prepared rubrics."

The construction of a mathematical model, especially for a complex problem, however, is an art and requires a keen mind coupled with a good mathematical and technological background. Examples of models will be presented later.

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8. Leontief, W.W., The Structure of the American Economy 1919-1939, second ed.; New York, Oxford University Press, 1951.
9. Dorfman, R., Application of Linear Programming to the Theory of the Firm; Berkeley; University of California Press, 1951.
10. Charnes, A., Cooper, W.W., and Henderson, A., An Introduction to Linear Programming; New York, John Wiley & Sons, 1953, Preface.

MATHEMATICS OF LINEAR PROGRAMMING

Before illustrating the application of linear programming to problems it is necessary that the reader understand certain basic material of vector geometry and matrix algebra. Only the minimum of information necessary to the solution of linear programming problems will be presented. Mathematical proofs and amplifying material can be found in the material listed in the bibliography, especially in Part II of An Introduction to Linear Programming by Charnes, Cooper and Henderson.[11]

Points and Vectors

The representation of a point in a plane by (x_1, y_1) where x_1 is the x coordinate (distance from the origin or zero point along the x axis), and y_1 is the coordinate in the y direction, is well known. Just as (x_1, y_1), (x_2, y_2) locate points in a plane, (x_1, y_1, z_1) and (x_2, y_2, z_2) locate points in the 3-dimensional space. (x_1, y_1, z_1), in addition to representing a point, may represent a vector which can be visualized as an arrow whose tail is at the origin and whose head is at the point (x_1, y_1, z_1).

Similarly, though less easily visualized, $(x_1, x_2, x_3, \ldots . x_n)$ represents a point or vector in an n-dimensional space, where x_1, x_2, through x_n are real numbers called the coordinates in the dimensions 1, 2, through n. Point or vector x may be represented either as $(x_1, x_2, \ldots, x_n)$, which is called a row vector,

or as $\begin{matrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{matrix}$, which is called a column vector.

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11. Ibid.

The subscript under x designates the coordinate meant. A superscript above identifies to which point one refers. Thus x^1 and x^2 means point 1 and point 2 respectively while x_2^3 means the second coordinate of point 3.

Two or more vectors in the same n-dimensional space may be added by adding corresponding coordinates to form a new vector. Thus, if $x^1 = (x_1^1, x_2^1, x_3^1)$ and $x^2 = (x_1^2, x_2^2, x_3^2)$ then x^3, the sum of x^1 and x^2 is presented as $x^3 = (x_1^3, x_2^3, x_3^3) = (x_1^1 + x_1^2, x_2^1 + x_2^2, x_3^1 + x_3^2)$ with $x_1^3 = x_1^1 + x_1^2$; $x_2^3 = x_2^1 + x_2^2$; $x_3^3 = x_3^1 + x_3^2$.

This addition of vectors may be illustrated in 2 dimensions as follows: Given: $x^1 = (2,1)$; $x^2 = (1, 3)$. Find x^3, the sum of x^1 and x^2.

$$\begin{aligned} x^3 &= x^1 + x^2 \\ &= (x_1^1 + x_1^2, x_2^1 + x_2^2) \\ &= (2 + 1 , 1 + 3) \\ &= (3, 4). \end{aligned}$$

The point $x^{(3)}$ may be found also by laying off from point $x^{(1)}$ a vector parallel to and the same length as vector $x^{(2)}$, as shown in Figure 1.

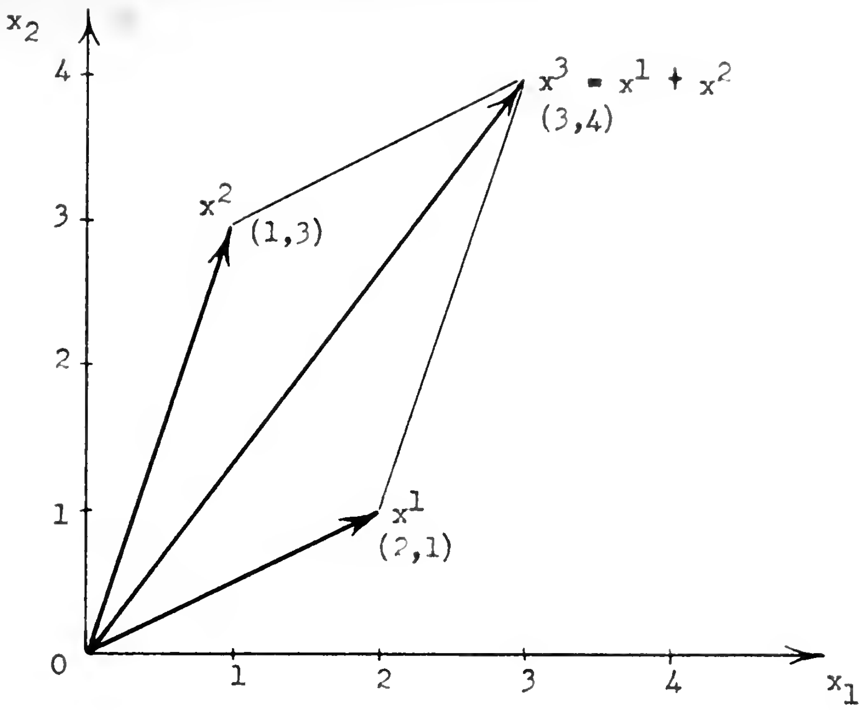

Fig. 1 Addition of Two Vectors

A point or vector may be multiplied by a real number. Thus if $x = (2, 1)$, then $2x = \left[(2)(2)\ ,\ (2)(1)\right] = (4, 2)$. Such a multiplication of a vector by a real number gives a new vector whose direction is exactly the same as the original vector but whose length is equal to the length of the original vector multiplied by the number.

A "unit vector" is a vector of value unity (one) in one axis only, and having value zero in every other axis. They are designated by $e^{(1)}$, $e^{(2)}$, $e^{(i)}$, $e^{(n)}$, etc., where the superscript designates which coordinate has value one. Thus in 2-dimensional space: $e^{(1)} = (1, 0)$ and $e^{(2)} = (0, 1)$ as shown in Figure 2.

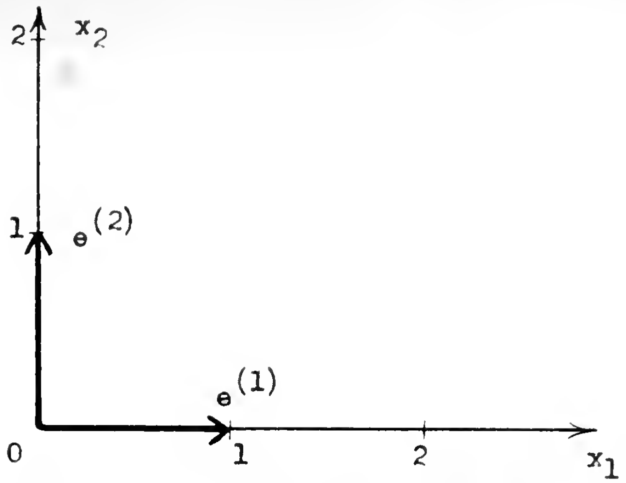

Fig. 2 Unit Vectors in 2-Dimensional Space

Similarly $e^{(i)}$ in an n-dimensional space is represented as: $e^{(i)} = (0, 0, .., 0, 1, 0, .., 0)$ where the 1 occurs in the i^{th} dimension.

Note that every vector in a 2-dimensional space (the plane) can be formed by a linear combination of the two unit vectors $e^{(1)}$ and $e^{(2)}$. For example: the vector $x^{(1)} = (3, 2)$ can be obtained from $(3)\ e^{(1)} + (2)\ e^{(2)} = 3\ (1, 0) + 2\ (0, 1) = (3, 2)$.

Similarly, every vector in an s-dimensional space can be formed by a suitable linear combination of the s unit vectors.

These s unit vectors in an s-dimensional space together form what is known as a linearly independent set.[12] This means that for a linear combination of the s unit vectors, $a_1\ e^{(1)} + a_2\ e^{(2)} + \ldots + a_s\ e^{(s)}$, to equal the zero vector $(0, 0, \ldots\ldots, 0)$, the coefficients of the e's must each equal zero. In 2 dimensions for example:

$$a_1\ e^{(1)} + a_2\ e^{(2)} = (0, 0)$$

$$= a_1\ (1, 0) + a_2\ (0, 1) = (0, 0)$$

$$= (a_1, 0) + (0, a_2) = (0, 0)$$

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12. Ibid. p. 48.

$= (a_1, a_2) = (0, 0)$ only if $a_1 = 0$ and $a_2 = 0$. $e^{(1)}$ and $e^{(2)}$ are thus a linearly independent set.

No vector in a linearly independent set can be formed from a linear combination of the other vectors in the set. For example, no linear combination of $e^{(1)}$ and $e^{(2)}$ can yield $e^{(3)}$.

This set of unit vectors is not the only linearly independent set. For example, in 2-dimensional space, (3, 1) and (2, 1) are linearly independent, for if $a(3, 1) + b(2, 1) = (0, 0)$, then

1. $3a + 2b = 0$ and
2. $a + b = 0$

From (2), $b = -a$, and substituting in (1), $3a - 2a = 0$ or $a = 0$ and thus $b = 0$.

Note, however, that in 2-dimensional space, (1, 3) and (2, 6) are not linearly independent but are linearly dependent. Each vector is a real number multiple of the other: $2(1, 3) = (2, 6)$.

An additional consequence of linearly independent sets is that if a point $P = a_1x^1 + a_2x^2 + \ldots + a_nx^n$ (where $x^1, x^2, \ldots, x^n$ is a linearly independent set), then the coefficients $a_1, a_2, \ldots, a_n$ are the only coefficients of the x^i's in terms of which one can express P as a sum of multiples of the x^i's, i.e., as a linear combination.

The maximum number of points (vectors) in a linearly independent set in 2-dimensional space is 2, in 3-dimensional space is 3, and in m-dimensional space is m. In an m-dimensional space, any linearly independent set of m points is called a basis. There can be any number of bases but each basis of an m-dimensional space must consist of

exactly m linearly independent points.

Linear Transformations

An important concept in linear programming is that of linear transformations. A transformation is a mapping or transfer of points from an n-dimensional space into a space of possibly different dimensions, each point assuming its new position in accordance with the directions specified by the particular transformation. For example, if the point $x = (1, 3)$ were transformed by T to the point $(2, 6)$ the transformation T of x equals (2) x indicated by $T(x) = 2x$.

A linear transformation is merely a restricted type of transformation and must conform to the following two requirements: [13]

1. $T(ax)$ must equal $aT(x)$. For example, if $x = (1, 3)$, $a = 2$, and T specifies a multiplication by 3

   $T(ax) = (3)(2)(1, 3) = (6, 18)$

   $aT(x) = (2)(3)(1, 3) = (6, 18)$ T thus conforms to the first requirement.

2. $T(x + w)$ must equal $T(x) + T(w)$.

   If T, again, specifies a multiplication by 3, $x = (1, 3)$ and $w = (2, 4)$, then

   $T(x + w) = 3(1 + 2, 3 + 4) = (9, 21)$

   $T(x) = 3(1, 3) = (3, 9)$ and

   $T(w) = 3(2, 4) = (6, 12)$ adding,

   $T(x) + T(w) = (3 + 6, 9 + 12) = (9, 21)$

   T thus conforms to the second requirement.

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13. Ibid. p. 44.

Because the set of m unit vectors in an m-dimensional space is a linearly independent set, or basis, every point in the m-dimensional space can be expressed as a linear combination of the m unit vectors. Thus, if x = (3, 0, 2), x can be expressed as

$$x = (3)\ e^{(1)} + (0)\ e^{(2)} + (2)\ e^{(3)} \text{ or}$$

$$x = \sum_{i=1}^{3} c_i\ e^{(i)} \text{ where } c_1 = 3;\ c_2 = 0;\ c_3 = 2$$

It follows therefore that if it is known what the linear transformation does to the unit vectors, the linear transformation of any point is known.

If $e^{(i)} = (0, \ldots, 1, 0, \ldots, 0)$ where the 1 occurs in the i^{th} dimension, and $i = 1, 2, \ldots, n$, are unit vectors in an n-dimensional space N, and $g^{(j)} = (0, \ldots, 1, 0, \ldots, 0)$, with $j = 1, 2, \ldots, m$, are unit vectors in the m-dimensional space M, then the linear transformation T of $e^{(i)}$ takes the form $T\ e^{(i)} = \sum_{j=1}^{m} a_{ij}\ g^{(j)}$. Thus: [14]

$$T\ e^{(1)} = a_{11}\ g^{(1)} + a_{12}\ g^{(2)} + \ldots + a_{1m}\ g^{(m)} = y^{(1)}$$

$$T\ e^{(2)} = a_{21}\ g^{(1)} + a_{22}\ g^{(2)} + \ldots + a_{2m}\ g^{(m)} = y^{(2)}$$

$$T\ e^{(k)} = a_{k1}\ g^{(1)} + a_{k2}\ g^{(2)} + \ldots + a_{km}\ g^{(m)} = y^{(k)}$$

$$T\ e^{(n)} = a_{n1}\ g^{(1)} + a_{n2}\ g^{(2)} + \ldots + a_{nm}\ g^{(m)} = y^{(n)}$$

The coefficients of the $g^{(j)}$'s above when written by themselves can be written in an array as follows:

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14. Ibid. p. 47.

$$T \sim \begin{matrix} a_{11} & a_{12} & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2m} \\ a_{k1} & a_{k2} & \cdot & \cdot & a_{km} \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nm} \end{matrix}$$

Such an array is customarily called a matrix of n rows and m columns. The matrix is said to be associated ($\sim$) with the linear transformation T. The entries in the k^{th} row are the coordinates of the point into which $e^{(k)}$ is carried by the transformation. Thus, $e^{(1)}$ which is the point (1, 0, 0,..., 0) in the n-dimensional space N is transformed into the point $y^{(1)}$ whose coordinates are $(a_{11}, a_{12}, .., a_{1m})$ in the m-dimensional space M.

Example:

N is a 3-dimensional space with
$e^{(1)} = (1, 0, 0)$; $e^{(2)} = (0, 1, 0)$; $e^{(3)} = (0, 0, 1)$
M is a 2-dimensional space with
$g^{(1)} = (1, 0)$; $g^{(2)} = (0, 1)$, and:

$$T \sim \begin{matrix} 2 & 1 \\ 1 & 2 \\ 2 & 3 \end{matrix}$$

Then

$$T \quad e^{(1)} = 2\, g^{(1)} + 1\, g^{(2)} = y^1 = (2,1)$$
$$T \quad e^{(2)} = 1\, g^{(1)} + 2\, g^{(2)} = y^2 = (1, 2)$$
$$T \quad e^{(3)} = 2\, g^{(1)} + 3\, g^{(2)} = y^3 = (2, 3)$$

Figure 3 illustrates these transformations from space N into space M.

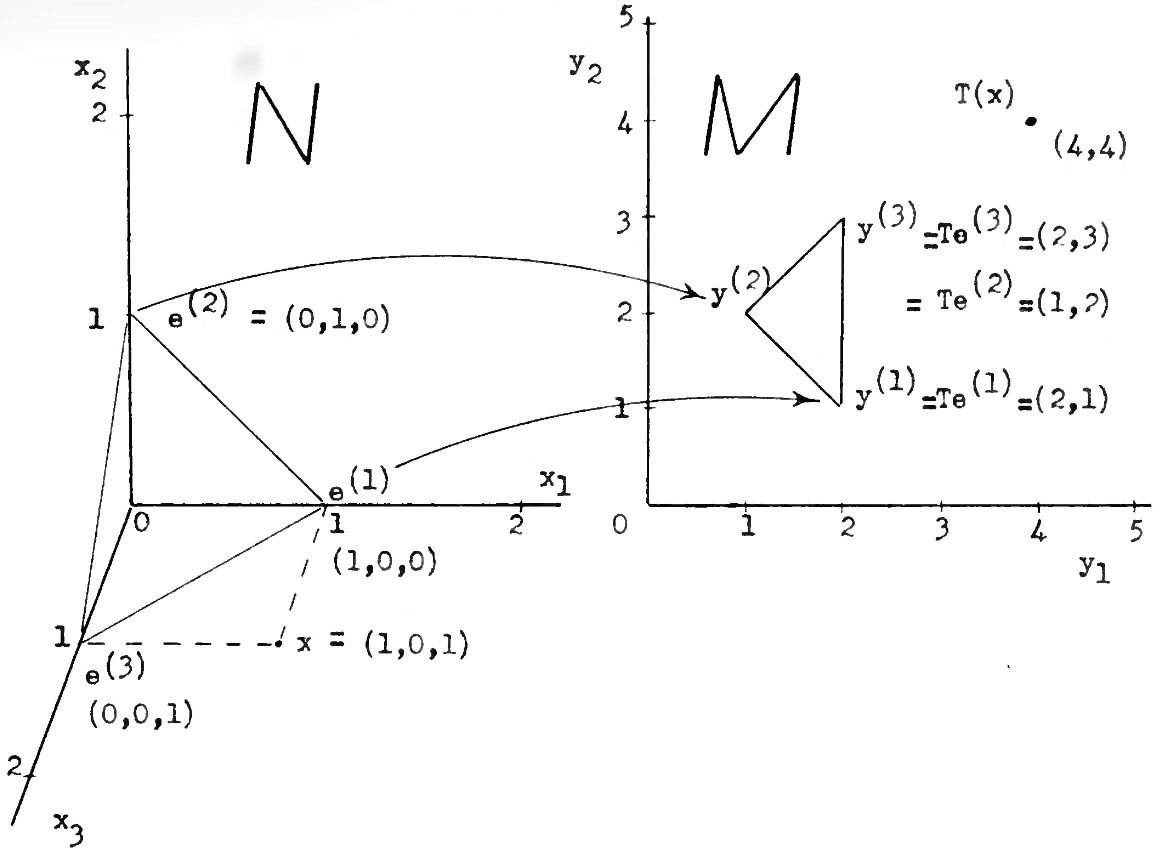

Fig. 3 Geometric Representation of Linear Transformations

Since it is known what the linear transformation T does to the unit vectors in N, what T does to each point x is also known. If

$$x = (x_1, x_2, x_3) = (1, 0, 1),$$

$$T\,(x) = x_1\, T\, e^{(1)} + x_2\, T\, e^{(2)} + x_3\, T\, e^{(3)}$$

$$= (1)\cdot(2, 1) + (0)\,(1, 2) + (1)\,(2, 3)$$

$$= (2, 1) + (2, 3) = \Big[(2 + 2), (1 + 3)\Big] = (4, 4)$$

Points x and T (x) are illustrated in Figure 3.

Convex Sets

A convex set is a collection of points such that, if x and y are any two points in the collection, every point on the line segment

joining x and y is also in the collection. Any point in a convex set which does not lie on the line segment joining some two other points in the set is known as an extreme point. (See Figure 4). A rectangle, for instance, is a convex set and the four corners are extreme points. A convex set which has a finite number of extreme points is given the special name of convex polyhedron.

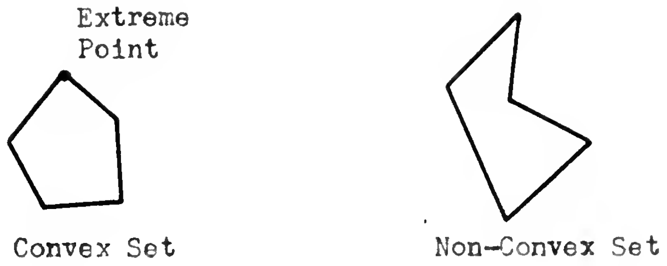

Fig. 4 Convex and Non-Convex Sets

In Figure 5 the convex polyhedron $P_1\ P_2\ P_3\ P_4$ is said to generate the cone $0\text{-}P_1\ P_2\ P_3\ P_4$. This means that those rays emanating from the origin 0 which pass through the perimeter of the convex polyhedron $P_1\ P_2\ P_3\ P_4$ form the sides of a cone. Such a cone is referred to as a convex polyhedral cone.

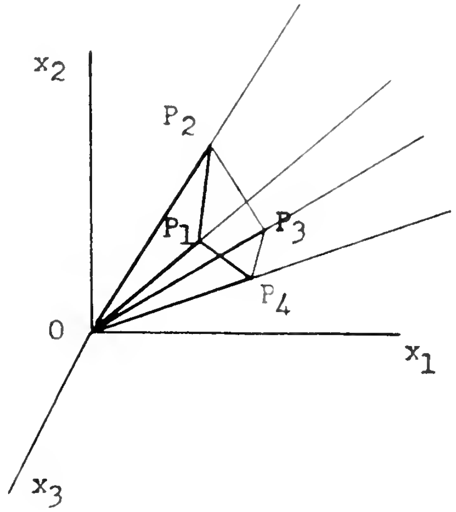

Fig. 5 Convex Polyhedral Cone

Clearly the unit vectors in an m-dimensional space generate a convex polyhedral cone which will encompass all positive vectors in the m-dimensional space.

The linear transformation of points has been discussed. Just as points in an n-dimensional space can be transformed into image points in an m-dimensional space, so can a convex polyhedron. Indeed, the image under (resulting from) a linear transformation of a convex polyhedron is a convex polyhedron. An illustration of this is shown in Figure 3. The convex polyhedron in space N formed by the points $e^{(1)}$, $e^{(2)}$, and $e^{(3)}$ (a triangle) has been transformed into a triangle formed by the points $y^{(1)}$, $y^{(2)}$, and $y^{(3)}$ in space M.

Simplex Method of Solution

The preceding mathematical discussion has presented some basic information necessary to the understanding of what a linear programming problem is and how it can be solved. A simple numerical example will

now be presented and its solution will be obtained by the simplex method.

The example consists of two inequations in three unknowns and a linear function (sometimes called a linear functional) of the three unknowns. The linear functional is to be maximized. A restriction is imposed that the unknowns when found must be non-negative, that is, must equal or be greater than zero (≥ 0).

$$2x + y + 2z \leq 10$$

$$x + 2y + 3z \leq 14$$

$$f(x, y, z) = x + y + z \quad \text{(maximum)}$$

$$\text{with } x \geq 0; \quad y \geq 0; \quad z \geq 0$$

To convert the inequalities to equations a non-negative variable is added to each inequation and the model appears as follows:

1. $$\begin{aligned} u + 2x + y + 2z &= 10 \\ v + x + 2y + 3z &= 14 \end{aligned}$$

2. $x, y, z, u, v, \geq 0$

3. $f(x, y, z) = x + y + z + (0)u + (0)v$ to be a maximum.

Stated in words the problem is to find the point or points in a 5-dimensional space U which have non-negative coordinates (u, v, x, y,z) satisfying equations (1) above and such that the linear functional ($x + y + z$) is a maximum. It can be shown that the point or set of points so described forms a convex set. The coefficients associated with the variables in equations (1) plus the right side of the equations yield 6 column vectors P_i in the 2-dimensional space W. Thus,

$P_1 = \begin{pmatrix}1\\0\end{pmatrix}$; $P_2 = \begin{pmatrix}0\\1\end{pmatrix}$; $P_3 = \begin{pmatrix}2\\1\end{pmatrix}$; $P_4 = \begin{pmatrix}1\\2\end{pmatrix}$; $P_5 = \begin{pmatrix}2\\3\end{pmatrix}$; and $P_0 = \begin{pmatrix}10\\14\end{pmatrix}$

Equations (1) can be restated:

$$\begin{array}{cccccccccccc} & P_1 & & P_2 & & P_3 & & P_4 & & P_5 & & P_0 \\ u & \begin{pmatrix}1\\0\end{pmatrix} & + v & \begin{pmatrix}0\\1\end{pmatrix} & + x & \begin{pmatrix}2\\1\end{pmatrix} & y & \begin{pmatrix}1\\2\end{pmatrix} & + z & \begin{pmatrix}2\\3\end{pmatrix} & = & \begin{pmatrix}10\\14\end{pmatrix}\end{array}$$

$$\text{or } uP_1 + vP_2 + xP_3 + yP_4 + zP_5 = P_0$$

Recall the discussion of linear transformations. The coefficients in the above equations form a matrix associated with the linear transformation which takes the point $Q = (u, v, x, y, z)$ in 5-dimensional space, U, into the point $P_0 = (10, 14)$ in 2-dimensional space, W.[15]

In the equations above, u is the coordinate along axis 1, v along axis 2, in the space U.

$$T\ e^{(1)} = P_1 = (1)\ g^{(1)} + (0)\ g^{(2)}$$
$$T\ e^{(2)} = P_2 = (0)\ g^{(1)} + (1)\ g^{(2)}$$
$$T\ e^{(3)} = P_3 = (2)\ g^{(1)} + (1)\ g^{(2)}$$
$$T\ e^{(4)} = P_4 = (1)\ g^{(1)} + (2)\ g^{(2)}$$
$$T\ e^{(5)} = P_5 = (2)\ g^{(1)} + (3)\ g^{(2)}$$

where the coefficients of the g's are the elements in the respective vectors P_i.

There is more than one point Q in U having non-negative coordinates u, v, x, y, z, which will satisfy the equations, i.e., satisfy $T(Q) = P_0$, and it can be shown that these values of Q form a convex set in U. For example: $Q = (u, v, x, y, z) =$

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15. Ibid. p. 53.

(2/3, 0, 0, 0, 14/3); Q = (10, 14, 0, 0, 0); and Q = (0, 0, 1/2, 0, 9/2), are each answers which will satisfy the two equations, as substitution back into the equations will show. All possible Q's which have non-negative coordinates u, v, x, y, and z form a convex set which, because it has a finite number of extreme points is a convex polyhedron.

An important theorem states that a linear functional f (u, v, x, y, z) defined on a convex polyhedron takes on its maximum (or minimum) at an extreme point of the convex set.[16] If the extreme points of the convex polyhedron generated by all values of Q which have non-negative coordinates could be found, the maximum value of the functional could also be found. That is to say, having the coordinates of an extreme point, and substituting them into the functional, a value of the functional would be obtained. Comparing this value of the functional with the value obtained from every other extreme point of the convex set, one could pick out the extreme point whose coordinates yielded the maximum value of the functional. These coordinates, or values for x, y, and z (u and v are of no interest, having been added merely to change the inequations to equations) are then the required values for the optimum solution of the problem.

The procedure, then, is to locate an extreme point, calculate the value of its functional, proceed to another extreme point, and so on, until the required maximum value of the functional is obtained. The extreme point which gives the maximum value of the functional is the solution.

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16. Ibid. P. 52.

Recalling the discussion of linear independence, it was stated that in 2-dimensional space the maximum number of points or vectors in a linearly independent set, or basis, is two. The above points P are all in 2-dimensional space and subjecting any pair of points, P, to the criterion for linearly independent points as described on page 9. it can be seen that any combination of two such points P is a linearly independent set. Any point, therefore, can be expressed as a linear combination of any two other such points P. For example, if x, y, and z were each equal to zero, the equation could be stated as

$$u\,P_1 + v\,P_2 + 0\,P_3 + 0\,P_4 + 0\,P_5 = P_0$$

or $u\,P_1 + v\,P_2 = P_0$. Inserting the coordinates of

P_1, P_2, and P_0

$u\,(1) + v\,(0) = 10$ or $u = 10$

$u\,(0) + v\,(1) = 14$ or $v = 14$

The point $Q = (\overset{u}{10}, \overset{v}{14}, \overset{x}{0}, \overset{y}{0}, \overset{z}{0})$ is an extreme point. This point Q fulfills the requirements of the equations of restrictions, but the value of the functional resulting from this point Q is not a maximum, but is, instead, zero. The simplex method is designed to permit movement from one extreme point to another and in such a direction that an increase in the functional will result.

To demonstrate the simplex method for the solution of this problem the P_i's are placed in a box arrangement. The P_0 has been moved into the first position.

Table 1

P_0	P_1	P_2	P_3	P_4	P_5
10	1	0	2	1	2
14	0	1	1	2	3

The original equations of the problem can be derived from the box arrangement as follows: The single vertical lines represent plus signs and the double lines separating P_0 from the rest of the arrangement represent equal signs. Visualizing the elements of each horizontal row as being multiplied by their respective unknowns and including the equal and plus signs one can recreate the two equations. Thus from row 1:

$$10 = u(1) + v(0) + x(2) + y(1) + z(2)$$

and from row 2:

$$14 = u(0) + v(1) + x(1) + y(2) + z(3)$$

An initial solution was obtained utilizing the basis set P_1, P_2. To show this, Table 1 is expanded into Table 2, where P_1 and P_2 have been placed to the left of P_0 in the stub column.

Table 2

$c \rightarrow$			0	0	1	1	1
$\downarrow$	Vector	P_0	P_1	P_2	P_3	P_4	P_5
0	P_1	10	1	0	2	1	2
0	P_2	14	0	1	1	2	3

Table 2 has the property of expressing any of the vectors in terms of the set P_1 and P_2. Any vector as designated by the column headings can be stated by multiplying each element in the column by

the vector on its row in the stub and adding. Thus for P_0 the element on the first line under P_0, 10, is multiplied by P_1 and added to the product of the second element, 14, with P_2.

Thus $P_0 = (10)\ P_1 + (14)\ P_2$

Similarly

$$\begin{aligned} P_1 &= (1)\ P_1 + (0)\ P_2 \\ P_2 &= (0)\ P_1 + (1)\ P_2 \\ P_3 &= (2)\ P_1 + (1)\ P_2 \\ P_4 &= (1)\ P_1 + (2)\ P_2 \\ P_5 &= (2)\ P_1 + (3)\ P_2 \end{aligned}$$

This follows from the fact that any point in a 2-dimensional space can be expressed as a linear combination of the points in a linearly independent set. Any of the above can be verified by inserting the numerical values of the P_i's. Thus for P_4

$$\underset{P_4}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} = (1) \underset{P_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + (2) \underset{P_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

The numbers in the top row and far left column designated by c are the coefficients of the corresponding variables in the functional whose maximum is sought. Thus the number, 1, above P_3, P_4, and P_5 is the coefficient in the functional of x, y, and z, respectively. The coefficient of u and v is 0 as shown by the 0 above and to the left of P_1 and P_2.

Table 2 completes the first set of calculations. All of the six vectors (P_1, P_2, P_3, P_4, P_5, and P_0) appearing at the top of the table have been solved or stated in terms of the basis P_1, P_2 and a feasible

solution is indicated by the elements under P_0; ie. $u = 10$, $v = 14$. Because P_3, P_4, and P_5 do not appear in the stub their corresponding variables x, y, and z are each equal to zero. The value of the functional at this stage is obtained by multiplying the feasible solution listed under P_0 by the functional coefficients listed to the left under c. The value of the functional is thus $(10)(0) + (14)(0) = 0$.

As was stated, it is desired to move to another extreme point to investigate the value of the functional. The simplex method provides an algorism which yields a new value of the functional that is no smaller than the previous value. That is to say, the new value of the functional will either be larger than the previous value or possibly the same value as the previous value of the functional but will never be smaller. The new extreme point found will increase the value of one of the variables x, y, or z at the expense of decreasing the present values of u and v namely 10 and 14, respectively. To find which variable to increase (what vector P_3, P_4, or P_5 to bring into the basis), the unit functional weights of x, y, and z are computed and compared. The largest value will designate the vector to bring into the basis.

For example, an increase in the size of x by one unit (from 0 to 1) will increase the size of the functional by 1, the functional coefficient of x. But from Table 2, $1\ P_3 = 2\ P_1 + 1\ P_2$ or an increase in x by 1 will decrease u by 2 and v by 1. The decrease in the functional caused by reduction of u and v is equal to $(0)(2) + (0)(1) = 0$ because the functional coefficients of u and v are each zero. A unit increase of x, therefore, will cause the functional to increase in value by 1.

Similarly, a unit increase of y will cause the functional value to increase by

$$\begin{array}{cccccccc} c_y & y & & c_u & u & & c_v & v \\ (1) & (1) & - & (0) & (1) & - & (0) & (2) = 1 \end{array}$$

The remaining variable z, if increased by 1, will likewise increase the functional by 1.

$$\begin{array}{cccccccc} c_z & z & & c_u & u & & c_v & v \\ (1) & (1) & - & (0) & (2) & - & (0) & (3) = 1 \end{array}$$

Since the unit functional weights of x, y, and z are equal, the value of the functional will be increased the same amount no matter which variable is increased (by inclusion of its P in the stub basis), hence it is immaterial which one is chosen. If the unit functional weights are not equal, the one with the largest positive value designates the P to include in the basis.

Table 2 provides a means for a systematic computation of the unit functional weights as will now be shown. The table is reproduced here for convenience as Table 3.

Table 3

c →			0	0	1	1	1
↓	Vector	P_0	P_1	P_2	P_3	P_4	P_5
0	P_1	10	1	0	2	1	2
0	P_2	14	0	1	1	2	3
F		0	0	0	0	0	0
C−F			0	0	1	1	1

The values in the row designated by F are obtained as follows: each element in a column is multiplied by the c which appears to the

far left in the element's row and the products are added to give the value of F for that column. Thus the value of F under P_0 is equal to $(10)(0) + (14)(0) = 0$; F under P_3 is equal to $(2)(0) + (1)(0) = 0$, etc. The F value under P_0 is the value of the functional obtained from the solution $Q = (\overset{u}{10}, \overset{v}{14}, \overset{x}{0}, \overset{y}{0}, \overset{z}{0})$. The F value in each of the other columns is the value of the decrease of the functional which would result from the decrease in value of the two variables u and v if a unit increase in the variable corresponding to that column were made. Thus, if x were increased from 0 to 1, the value of $F_x = 0$ indicates that the reductions of u and v would have no effect upon the value of the functional.

The elements in the row C-F are the unit functional weights, and are obtained by subtracting the F value in each column from the corresponding C value at the top of the column.

The unit functional weights (C-F) are examined. The largest positive value determines which vector P to introduce into the basis in the stub to replace one P which is now in the stub. This replacement of an existing P in the basis by a new one corresponds to moving from one extreme point to another. By introducing the P which has the largest unit functional weight (C-F) the largest increase in the value of the functional is insured. As long as there are positive values of (C-F) an increase in the functional is possible.

In the present problem there are three equal positive values of (C-F), each equal to 1. It is immaterial which one is chosen though the length of the computations might be shortened somewhat by a judicious choice. For illustration, P_5 is chosen to replace one of the

vectors, P_1 or P_2, in the stub.

The choice of which vector P_1 or P_2 in the stub to be replaced by the incoming vector P_5 is made as follows: Divide each positive element in the P_5 column into the corresponding element in the P_0 column and compare the results. Thus

$$\frac{10}{2} = 5$$

$$\frac{14}{3} = 4\ 1/3$$

The smallest result, here 4 1/3, designates the vector P_2 of the stub as the one to be replaced by P_5. 4 1/3 or (14/3) is the new value of z. Since the vector P_2 is being replaced, its variable v decreases in value from 14 to 0. A blank table similar to Table 3 is now laid out but with P_2 in the stub replaced by P_5. Note that the c = 1 of P_5 is placed to the left of P_5 in the stub of Table 4.

Table 4

c→			0	0	1	1	1
↓	Vec-tor	P_0	P_1	P_2	P_3	P_4	P_5
0	P_1						
1	P_5						

To complete the table, one recalls that a property of this arrangement is that any vector as listed at the head of a column can be expressed as a linear combination of the two vectors in the stub. If the vectors P_1, P_2, P_3, P_4, P_5, can be expressed as linear combinations of P_1 and P_5, the coefficients will be the elements of the table. From Table 3 the following equations hold:

$$
\begin{aligned}
&(a) \quad P_0 = 10\,P_1 + 14\,P_2 \\
&(b) \quad P_1 = 1\,P_1 + 0\,P_2 \\
&(c) \quad P_2 = 0\,P_1 + 1\,P_2 \\
&(d) \quad P_3 = 2\,P_1 + 1\,P_2 \\
&(e) \quad P_4 = 1\,P_1 + 2\,P_2 \\
&(f) \quad P_5 = 2\,P_1 + 3\,P_2
\end{aligned}
$$

From equation (f) $P_2 = -2/3\ P_1 + 1/3\ P_5$ thus $-2/3$ and $1/3$ are the elements in the new table under P_2. Substituting $-2/3\ P_1 + 1/3\ P_5$ into equation (a)

$$P_0 = 10\,P_1 + 14\,(-2/3\,P_1 + 1/3\,P_5)$$

or $P_0 = 2/3\,P_1 + \frac{14}{3}\,P_5$. Thus $2/3$ and $14/3$ are the elements to be filled in under P_0.

Similarly,

$$
\begin{aligned}
P_1 &= 1\,P_1 + 0\,P_5 \\
P_3 &= 4/3\,P_1 + 1/3\,P_5 \\
P_4 &= -1/3\,P_1 + 2/3\,P_5 \\
P_5 &= 0\,P_1 + 1\,P_5
\end{aligned}
$$

determine the elements for the other columns. The values of F and C-F are determined in the same manner as employed in Table 3. The completed table, Table 5 then appears as follows:

Table 5

$c \rightarrow$			0	0	1	1	1
$\downarrow$	Vector	P_0	P_1	P_2	P_3	P_4	P_5
0	P_1	2/3	1	-2/3	4/3	-1/3	0
1	P_5	14/3	0	1/3	1/3	2/3	1
F		14/3	0	1/3	1/3	2/3	1
C - F			0	-1/3	2/3	1/3	0

Table 5 shows (in the P_0 column) that a new solution $u = 2/3$, $z = 14/3$, with $x = 0$, $y = 0$, $v = 0$, has been obtained. The F value below P_0 shows that the value of the functional has been increased from 0 to 14/3, and the two positive values in the C-F row show that further improvement is possible. The 2/3 value of C-F in column P_3 shows that P_3 is the most profitable vector to introduce into the basis. As before the division of the elements under P_0 by those under P_3 shows that P_1 is the vector in the basis to be replaced by P_3. Carrying out the techniques employed previously Table 6 is constructed using the equations obtained from Table 5 to compute the elements in Table 6.

Table 6

$c \rightarrow$			0	0	1	1	1
$\downarrow$	Vector	P_0	P_1	P_2	P_3	P_4	P_5
1	P_3	1/2	3/4	-1/2	1	-1/4	0
1	P_5	9/2	-1/4	1/2	0	3/4	1
F		5	1/2	0	1	1/2	1
C - F			-1/2	0	0	1/2	0

Examination of Table 6 reveals a new feasible solution for Q, namely, Q = (0, 0, 1/2, 0, 9/2), and an increased value of the functional, namely 5. The positive value of C-F, the 1/2 in column P_4, indicates that a larger value of the functional can be obtained by introducing P_4 into the basis. Because 3/4 in row P_5 is the only positive value in the P_4 column, P_5 is the vector in the basis to be replaced. Carrying through the same procedure as before by computing the elements for the next table from the equations indicated in Table 6, the final table, Table 7 is completed

Table 7

$c \rightarrow$			0	0	1	1	1
$\downarrow$	Vector	P_0	P_1	P_2	P_3	P_4	P_5
1	P_3	2	2/3	-1/3	1	0	1/3
1	P_4	6	-1/3	2/3	0	1	4/3
F		8	1/3	1/3	1	1	5/3
C - F			-1/3	-1/3	0	0	-2/3

The non-appearance of any positive values of C-F, the unit functional weights, indicates that no further increase in the value of the functional is possible. Reading in the P_0 column it is seen that the optimum solution is $x = 2$, $y = 6$, $u = 0$, $v = 0$, $z = 0$ and the maximum value possible of the functional is 8.

The procedure for arriving at the table elements, namely by solution of equations from the preceding table, becomes extremely laborious when the number of variables and the number of equations is increased. A method exists that permits easy calculation of the elements by a systematic mechanical method. This method permits the solution of extremely large problems involving many variables and many equations in the original problem.

The sample problem which has been solved originated as a set of two inequations. By introducing the additional unknowns to convert the inequations to equations the vectors $P_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $P_2 = \begin{pmatrix}0\\1\end{pmatrix}$ were introduced. These two slack vectors, as they are called, provided a known basis from which the first feasible solution ($u = 10$, $v = 14$) was obtained. When the size of the problem is increased the introduction of such a simple basis becomes of more importance. If the original set of restrictive conditions are equations instead of inequations, slack vectors, now known as artificial vectors, may still be added to provide a basis. The associated variables, however, must be included in the functional to be maximized and each such variable must have a negative coefficient of sufficient size so as to insure that the variable will not appear in the final answer.

The above sample problem was concerned with the maximization of

a linear functional. As was stated previously the optimization in a linear programming problem may be the minimization of some function such as cost. A minimization problem may be solved in a similar fashion except that successive extreme points are found by an algorism which insures that each new value of the functional that is obtained will be no larger than any previous one.

ASSUMPTIONS, POSSIBILITIES AND LIMITATIONS

The bibliography contains accounts of the application of linear programming to industrial problems. In the main, however, the practicality of the technique suffers from certain restrictive assumptions, difficulties of model building, lack of proper data, necessity for complex computing machines and the necessity that a linear programming program be set up, if not operated, by highly trained personnel. The last two may impose severe financial restrictions.

The assumptions that must be met in order to apply linear programming are as follows:[17]

1. The assumption of linearity. A linear process is one whose ratio of output to input is constant.[18] Thus for an operating machine tool, if an input of one unit of raw material produces one unit of output in a unit of time, an input of 100 units must produce 100 units of output in 100 units of time. Such an assumption neglects such aspects as rejects, tool wear, delay in delivery of material, set-up time, industrial fatigue, or unauthorized absence of the operator, etc. Such non-linearity of a production process is, of course, nothing new and confronts production control personnel no matter what method they employ. Because of the large number of computations made in linear programming, however, and the fact that the results or decisions may

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17. Dorfman, R., Application of Linear Programming to the Theory of the Firm; Berkeley, University of California Press, 1951, p. 80.
18. This ratio is a result of the technical aspects of the specific process or activity. In the machining of metal, for instance, specified speeds and feeds in addition to other specifications determine the process. If the speed or feed were changed a technically feasible process might result. Such a modification of a process, however, is, in linear programming terms , a new and distinct process.

affect a large segment of a firm's production program it becomes necessary to insure a high degree of accuracy. Morgenstern states:

> Linear programming, or any other similar utilization of great masses of economic data, cannot be expected to make decisive practical progress until there is satisfaction that the data warrant the implied extensive and costly numerical operations.[19]

Other examples of non-linearity are the reduction of sales price per unit for volume sales and the rise in productivity resulting from minor technological improvements.

2. The assumption of divisibility. Any process may be operated at any positive level. This assumes, for instance, that if one large press can stamp out 100 fenders in an hour then one half of a press can stamp out 50 fenders in an hour.

3. The assumption of additivity:

> It is assumed that two or more processes can be used simultaneously, within the limitations of available resources, and that if this is done the quantities of the outputs and inputs will be the sums of the quantities which would result if the several processes were used individually.[20]

given conditions

4. The assumption that the number of processes is finite. Because of the restricted field of interest of many firms the processes or activities are indeed finite. Some industries, however, such as agriculture, oil refining, or the chemical industries have an infinite range of choices.

Melvin E. Salveson has listed some of the problems which can and

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19. Morgenstern, O., "The Accuracy of Economic Observations," <u>Activity Analysis of Production and Allocation</u>, Cowles Comm. for Research in Economics, Monograph 13; New York, John Wiley and Sons, 1951, p. 283.
20. Dorfman, R., <u>Application of Linear Programming to the Theory of the Firm</u>; Berkeley, University of California Press, 1951, p. 81.

cannot be solved by linear programming.[21] He states that this technique can be used:

1. to determine the optimum shop load during any one time period or over several time periods.
2. to determine the optimum amount of overtime to use on any item or machine tool.
3. to determine the optimum mix of commodities to make in the shop or factory.
4. to determine the amount of rerouting of work in the shop.
5. to determine the optimum level of inventory (raw, in-process, and finished).
6. to determine the optimum distribution of production of commodities between time periods.

He states that linear programming cannot be used:

1. to determine a schedule for a shop.
2. to select the best sequence of production in order to minimize, say, set up time.
3. to eliminate need for expediters.
4. to avoid all conflicts in production, such as temporary bottlenecks and hence, its program is not necessarily always achievable (unless it is constructed with overly liberal delay allowances).
5. to give delivery dates more precisely than the length of its time periods.

The reasons for the non-applicability of linear programming to the five items above are as follows:

(a) Item 1 and item 5. These refer to the scheduling of activities. Production or any other activity of a linear programming problem when considered in relation to time is considered to begin at the start of a specified time period, continue during the time period, at a constant rate and to reach completion at the end of the time period. This activity time requirement is faced in machine or shop loading problems also but suitable assumptions or restrictions in the formulation of the

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21. Salveson, M.E., "Mathematical Models in Management Programming," Journal of Industrial Engineering, March 1954, vol. 5, No. 2, p. 10.

mathematical model may allow an operationally meaningful solution. An example of such a restrictive assumption will be given later.

(b) Item 2. The selection of the sequence of production is determined by the technology of the production processes. Thus, for example, the manufacture of a certain item might require that a milling operation precede a drilling operation. This sequence is part of the technological requirements for the manufacture of the item. In addition , set up time is a non-linear activity or function of a process. Thus one hour of set up time may be required for an operation regardless of the level of activity of the process.

(c) Item 3 and item 4. These restrictions on the applicability of linear programming to production result from the inability of any mathematical model to account fully for the emergencies of the every day world. If every exigency were known ahead of time it might possibly be incorporated in the model. The possibility of forecasting the occurence of every conceivable perturbation or disturbance in the orderly activities of nature is, of course, extremely slim. Many conflicts in production, such as machine breakdown or non-delivery of raw material due to personnel error, may be compensated for by constructing the model with very liberal delay allowances. Expediters are necessary to keep the actual production as close to the program as possible.

MODELS

Many mathematical models have been presented in linear programming literature and work in the area of model building is proceeding. To show some of the range of possibilities a few models are presented here.

Machine Assignment[22]

A number of operations are to be performed in each of a number of mutually exclusive operation types. A number of machines of various types are available. The productivity of each machine type in each of the various types of operations is known.

Required: to assign the machines to operations such that the total productivity of all machines on all operations is a maximum.

Let:

m = the number of types of machines.

a_i = the number of machines of type (i) where $i = 1, 2, 3, \ldots, m$.

n = the number of types of operations.

b_j = the number of operations of type (j) where $j = 1, 2, 3, \ldots, n$.

c_{ij} = the productivity of a machine of type (i) in performing operation (j).

x_{ij} = the number of type (i) machines to be assigned to a type (j) operation.

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22. Harrison, J.O., Jr., Linear Programming and Operations Research, Informal Seminar in Operations Research, Paper No. 2; Baltimore, Johns Hopkins Univ., October 13, 1953, pp. 1?-13.

The x_{ij} are to be selected such that:

$$x_{ij} \geq 0$$

$$\sum_{j=1}^{n} x_{ij} \leq a_i$$

$$\sum_{i=1}^{m} x_{ij} = b_j$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \; x_{ij} = \text{maximum}$$

Production Line Loading to Meet Sales Requirements[23]

Given: one production line producing one type of product which sells for a fixed unit price. The unit costs of regular time production, overtime production, and storage are known. The rates of production per unit time are known. A sales demand during each of a number of successive time periods is known. The time period is one month.

Desired: a production loading plan which will meet the sales demand and minimize the combined costs of production and storage. Inventory is taken at the end of each month.

Let:

k = number of time periods (months) to load.

s_i = number of units of finished product to be sold during the i^{th} period.

I_0 = initial inventory.

R = maximum number of units which can be produced during a month on regular time.

U = maximum number of units which can be produced during a month on overtime.

C_I = cost of storage of one unit of product for one month.

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23. Ibid. pp. 14-15.

C_x = unit production cost utilizing regular time.

C_y = unit production cost utilizing overtime.

Find: .

x_i = number of units to be produced on regular time during the i^{th} month.

y_i = number of units to be produced on overtime during the i^{th} month.

(a) $x_i \, , \; y_i \geq 0 \qquad (i = 1, 2, \ldots\ldots, k)$

(b) $x_i \leq R$

(c) $y_i \leq U$

(d) $\sum_{a=1}^{i} (x_a + y_a) \geq \sum_{a=1}^{i} s_a - I_0$

(e) $\sum_{i=1}^{k} (C_I I_i + C_x x_i + C_y y_i)$ to be minimized, where

(f) $I_i = I_0 + \sum_{a=1}^{i} (x_a + y_a - s_a)$

and $(a = 1, 2, \ldots\ldots, i)$

Equation (d) is obtained as follows:

The inventory during any month, a, is denoted by I_a. Since the inventory at the end of any month is equal to the inventory at the end of the previous month plus the monthly production minus the monthly sales,

$$I_a = I_{a-1} + x_a + y_a - s_a$$

Summing this equation from a = 1 to a = i yields equation (f). Since the right side of equation (f) must be equal to or greater than zero (≥ 0), or

$$I_0 + \sum_{a=1}^{i} (x_a + y_a) - \sum_{a=1}^{i} s_a \geq 0$$

A slight rearrangement yields equation (d).

A Numerical Example - Product Assembly Over Consecutive Time Periods[24]

To illustrate the complexity to which even problems of small size are subject a numerical example is presented.

Assumptions: a factory making three products, each of which can be final products and two of which are in addition intermediate products.

Product 1 (P_1) is an assembly of one unit each of P_2 and P_3.

Product 2 (P_2) is composed of 2 units of Product 3 (P_3).

There are two types of operations, type 1 (T_1), and type 2 (T_2).

One hour of T_1 is required for one unit of P_1.

One hour of T_1 plus one hour of T_2 is required for one unit of P_2.

Two hours of T_2 are required for one unit of P_3.

The above assumptions are summarized in the following table:

	To make one unit P_1	To make one unit P_2	To make one unit P_3
P_1 (unit)			
P_2 (unit)	1		
P_3 (unit)	1	2	
T_1 time (hours)	1	1	
T_2 time (hours)		1	2

24. Jackson, J.R., "Mathematical Models with Examples from Linear Programming," Industrial Logistics Research Project Report No. 4; Los Angeles, Univ. of California.

Further assumptions:

3000 hours per week of T_1 are available.

4000 hours per week of T_2 are available.

3000 units of P_3 are in storage at the commencement of the production period.

Net profit from the sale of:

$P_1 = \$14$

$P_2 = \$\ 5$

$P_3 = \$\ 1$

The net increase in profit from the sale of:

$P_1 = 14 - 5 - 1 = \$8$ and of

$P_2 = 5 - 2\,(1) = \$3$

Items produced during any given week cannot enter into further production until the next week. ie. units of P_2 made during the first week cannot be assembled into P_1 until the second week.

Requirement: to operate for three weeks so as to maximize profits.

Let:

x^t = production of P_1 in week t: $t = 1, 2, 3.$

y^t = " " P_2 in week t: $t = 1, 2, 3.$

z^t = " " P_3 in week t; $t = 1, 2, 3.$

u^t = number of units of P_1 stored during week t; $t = 1, 2, 3.$

v^t = " " " " P_2 " " " "; $t = 1, 2, 3.$

w^t = " " " " P_3 " " " "; $t = 1, 2, 3.$

a^t = number of hours of idle time of T_1 in week t; $t = 1, 2, 3.$

b^t = number of hours of idle time of T_2 in week t; $t = 1, 2, 3.$

From the table it can be seen that:

0 = number of units of P_1 consumed in week t; $t = 1, 2, 3.$

x^t = " " " " P_2 " " " "; $t = 1, 2, 3.$

$x^t + 2y^t$ = " " " " P_3 " " " "; $t = 1, 2, 3.$

In addition:

0 = number of units of P_1 available at beginning of week 1.

0 = " " " " P_2 " " " " " ".

3000 = " " " " P_3 " " " " " ".

$u^t + x^t$ = P_1 available at beginning of week $(t + 1)$; $t = 1, 2, 3.$

$v^t + y^t$ = P_2 " " " " " $(t + 1)$; $t = 1, 2, 3.$

$w^t + z^t$ = P_3 " " " " " $(t + 1)$; $t = 1, 2, 3.$

From the above restrictions the following equations can be obtained:

$$
\begin{array}{lll}
 & u^1 = 0 & \\
 & v^1 + x^1 = 0 & \\
 & w^1 + x^1 + 2y^1 = 3000 & \\
\text{I.} & u^t = u^{t-1} + x^{t-1}; & t = 2, 3. \\
 & v^t + x^t = v^{t-1} + y^{t-1} & t = 2, 3. \\
 & w^t + x^t + 2y^t = w^{t-1} + z^{t-1}; & t = 2, 3.
\end{array}
$$

Availability of machine time leads to the following equations:

$$
\begin{array}{lll}
\text{II.} & x^t + y^t + a^t = 3000 & t = 1, 2, 3. \\
 & y^t + 2z^t + b^t = 4000 & t = 1, 2, 3.
\end{array}
$$

In addition:

$$\text{III. } x^t, y^t, z^t, u^t, v^t, w^t, a^t, b^t, \geq 0; \qquad t = 1, 2, 3.$$

Finally the net profit:

$$\text{IV. } P = 8(x^1 + x^2 + x^3) + 3(y^1 + y^2 + y^3) + (z^1 + z^2 + z^3)$$

Thus, to operate for three weeks so as to maximize profits calls for maximizing IV subject to the restrictions imposed by I, II, and III.

The preceding problem by J.R.Jackson can now be solved in the

following way:

Substituting the values for (t) the following equations are obtained:

$$u^1 = 0$$

$$v^1 + x^1 = 0$$

$$w^1 + x^1 + 2y^1 = 3000$$

$$u^2 = u^1 + x^1$$

$$u^3 = u^2 + x^2$$

$$v^2 + x^2 = v^1 + y^1$$

$$v^3 + x^3 = v^2 + y^2$$

$$w^2 + x^2 + 2y^2 = w^1 + z^1$$

$$w^3 + x^3 + 2y^3 = w^2 + z^2$$

$$x^1 + y^1 + a^1 = 3000$$

$$x^2 + y^2 + a^2 = 3000$$

$$x^3 + y^3 + a^3 = 3000$$

$$y^1 + 2z^1 + b^1 = 4000$$

$$y^2 + 2z^2 + b^2 = 4000$$

$$y^3 + 2z^3 + b^3 = 4000$$

Since $u^1 = 0$ it may be eliminated, and since $v^1 = -x^1$ and $u^2 = x^1$, v^1 and u^2 may be replaced by $-x^1$ and x^1 respectively. The result is a set of 12 equations as follows:

1. $w^1 + x^1 + 2y^1 = 3000$
2. $u^3 - x^1 - x^2 = 0$
3. $v^2 + x^2 + x^1 - y^1 = 0$
4. $v^3 + x^3 - v^2 - y^2 = 0$
5. $w^2 + x^2 + 2y^2 - w^1 - z^1 = 0$
6. $w^3 + x^3 + 2y^3 - w^2 - z^2 = 0$

7. $x^1 + y^1 + a^1 = 3000$

8. $x^2 + y^2 + a^2 = 3000$

9. $x^3 + y^3 + a^3 = 3000$

10. $y^1 + 2z^1 + b^1 = 4000$

11. $y^2 + 2z^2 + b^2 = 4000$

12. $y^3 + 2z^3 + b^3 = 4000$

These 12 equations are displayed in the familiar arrangement of Table 8. It appears that it will be possible to rearrange the table so as to provide twelve linearly independent vectors to form the initial basis. The below listed steps will yield an equivalent matrix which will have the desired form.

1. Replace row 6 by the sum of row 6 and row 5.
2. Interchange columns v^3 and w^1.
3. Replace row 4 by the sum of row 4 and row 3.
4. Replace row 5 by the sum of row 5 and row 1.
5. Replace row 6 by the sum of row 6 and row 1.
6. Interchange columns v^2 and w^1.
7. Interchange columns u^3 and w^1.

Moving the basis to the left side of the table, adding the functional coefficients, the stub column, and the F and C − F rows completes the first set of calculations as shown in Table 9. Succeeding tables can then be utilized to carry out the steps of the simplex method.

Table 8

Equation	P_0	x^1	x^2	x^3	y^1	y^2	y^3	z^1	z^2	z^3	u^3	v^2	v^3	w^1	w^2	w^3	a^1	a^2	a^3	b^1	b^2	b^3
(1)	3000	1			2									1								
(2)	0	-1	-1								1											
(3)	0	1	1		-1							1										
(4)	0			1		-1						-1	1									
(5)	0		1			2		-1						-1	1							
(6)	0			1			2		-1						-1	1						
(7)	3000	1			1												1					
(8)	3000		1			1												1				
(9)	3000			1			1												1			
(10)	4000				1			2												1		
(11)	4000					1			2												1	
(12)	4000						1			2												1

Table 9

c															8	8	8	3	3	3	1	1	1
	Vector	P_0	w^1	u^3	v^2	v^3	w^2	w^3	a^1	a^2	a^3	b^1	b^2	b^3	x^1	x^2	x^3	y^1	y^2	y^3	z^1	z^2	z^3
	w^1	3000	1												1			2					
	u^3			1											-1	-1							
	v^2				1										1	1		-1					
	v^3					1									1	1	1	-1	-1				
	w^2	3000					1								1	1		2	2		-1		
	w^3	3000						1							1	1	1	2	2	2	-1	-1	
	a^1	3000							1						1			1					
	a^2	3000								1						1			1				
	a^3	3000									1						1			1			
	b^1	4000										1						1			2		
	b^2	4000											1						1			2	
	b^3	4000												1						1			2
F																							
C - F															8	8	8	3	3	3	1	1	1

To show the amount of computation required to solve this problem by an automatic computer recall that advancing from one table to the next is in effect the movement from one extreme point to another. To go from one extreme point to another by the following steps requires approximately the number of computer operations as indicated for each step.

1. the calculation of the unit functional weights (C - F). 286 operations.
2. the inspection and comparison of the unit functional weights and the selection of the one with the largest positive value. This determines which vector is to be placed in the basis 36 operations.
3. The selection of the basis vector which is to be replaced by the incoming vector. 60 operations.
4. the calculation of the new value of the functional. 2 operations
5. The determination of the column vectors in terms of the new basis. 242 operations.

The total number of operations required then to advance from one extreme point to the next is 626.

Computational experience indicates that, in general, the number of extreme points which must be examined is approximately twice the number of equations. In this problem then, the total number of computer operations is (2)(12)(626) = 15024.

A Card Programmed Calculator (C.P.C.) which is a general purpose floating point digital computer that can perform 100 operations per

minute would then take approximately two and one-half hours of actual computer operation to solve this problem. This computer, however, cannot store enough information to allow continuous operation. The manual bookkeeping required for the removal and re-insertion of information doubles or triples the time required, and this problem, therefore, would require between five and seven and one half hours for solution.

CONCLUSION

Linear programming is thus a mathematical technique to find the values of non-negative variables or unknowns which are subject to linear restrictions and such that the values will optimize a linear functional. It has been shown that the linear programming problem can be visualized as a geometric representation of vectors or points in imaginary spaces of many dimensions. The solution of a simple numerical example has been demonstrated using the simplex method.

The requirements that must be met before this technique can be used for practical problems have been discussed. It can thus be seen that linear programming has a restricted usefulness for Industrial Engineering applications. As Joseph O. Harrison says:[25]

> The practical difficulties encountered in applying linear programming are threefold: (1) the expression of realistic objectives and constraints in measurable terms, (2) the determination of suitable numerical values for coefficients, and (3) the computational labor required to numerically execute large linear programming problems.

Like any mathematical technique, linear programming must not be considered a panacea for all the problems of Industrial Engineering. It is restricted in the scope of its application both by the nature of the problem and by the economic considerations of computation.

The history of the application of linear programming to practical problems covers too short a period of time and the application has been too limited to permit a complete evaluation of the usefulness of this technique to Industrial Engineering. It can be expected, however, that

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25. Harrison, J.O., Jr., Linear Programming and Operations Research, Informal Seminar in Operations Research, Paper No. 2; Baltimore, Johns Hopkins Univ., October 13, 1953, p. 18.

the considerable amount of study now being devoted to both the theory and the practical aspects of linear programming will result in the overcoming of many of the difficulties limiting its usefulness, and that Industrial Engineering will make more and more use of this technique in the future.

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